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ON ARBITRARY SEQUENCES OF ISOMORPHISMS IN $R^m \to R^m$

BY

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ABSTRACT. In this paper a new, clean proof of an algebraic theorem needed in ordinary differential equations is presented. The theorem involves the existence and uniqueness of a "complete splitting" for some subsequence of an arbitrary sequence of isomorphisms of Euclidean m-space. In the positive-definite case, a complete splitting is a limit condition on eigenspaces and eigenvalues.

1. **Introduction.** One of the hardest parts of the C^1 closing lemma [2] is the proof of the algebraic "subsequence and decomposition theorem". As remarked there, it is hard because vector techniques are used in place of subspace techniques. In this paper, a simpler proof of the theorem is given. A uniqueness theorem, new here, is also proved. Thanks are due to G. Mostow and D. Anosov for useful conversations.

We now state the main theorems, see §2 for the definitions of the terms employed and §§5, 6 for the proofs.

Let H be an inner product space and let $\{T_k\}$ be an arbitrary sequence of monomorphisms $R^m \to H$.

Existence theorem. *There exists a subsequence $\{T_{k_n}\} \subset \{T_k\}$ having a complete splitting.*

Uniqueness theorem. *If $R^m = \bigoplus W^j$ is a complete splitting for $\{T_k\}$ and $R^m = \bigoplus Y^j$ then $\bigoplus Y^j$ is a complete splitting for $\{T_k\}$ iff the flags of $\bigoplus Y^j$ and $\bigoplus W^j$ are equal.*

2. **Definitions and notations.** Let G, H be Euclidean spaces and call $M(G, H)$ the set of all monomorphisms $G \to H$.

Definition. For $T \in M(G, H)$, $m(T) = \min\{|Tx|: x \in G, |x| = 1\}$, $\|T\| = \max\{|Tx|: x \in G, |x| = 1\}$, and $\operatorname{bol}(T) = \|T\|/m(T)$. These numbers are called the minimum norm of T, the norm of T, and the bolicity of T.

Clearly $m(T) \neq 0$ for $T \in M(G, H)$. The bolicity measures how much T distorts the unit ball of G. The larger $\operatorname{bol}(T)$ is, the greater the distortion. Families

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of $T \in M(G, H)$ having uniformly bounded bolicity are easy to deal with, though not precompact.

Notation. If $G = \bigoplus G^j$ then ${}^jG = \bigoplus_{i \neq j} G^i$. The symbol $\oplus$ denotes the usual direct sum, $\bigoplus$ denotes orthogonal direct sum.

Definition. The flag of $G = \bigoplus G^j$ is the ascending sequence of subspaces: $G^1, G^1 \oplus G^2, \cdots, G^1 \oplus \cdots \oplus G^{L-1}, G$.

Definition. If $T \in M(G, H)$ and $\bigoplus G^j = G$, the perpendicular transform of T is $\hat{T}: G \to H$ by $\hat{T} = \Sigma A^j$ where $A^j: G^j \to [T({}^jG)]^{\perp}$ is defined by the commutativity of

$$\begin{tikzcd} G \arrow[r, "T"] & H \arrow[d, "\pi^j"] \\ G^j \arrow[u, hook] \arrow[r, "A^j"] & {[T({}^jG)]^{\perp}} \end{tikzcd}$$

π^j being the orthogonal projection.

Definition. The maps A^j are called the altitudes of T respecting $\bigoplus G^j$.

It is an open problem to investigate the algebraic properties of $\hat{}$. It is clear that $\hat{T} \in M(G, H)$ but that $\hat{}$ is not linear.

Definition. Let $\{T_n\}$ be a sequence in $M(G, H)$. Then $G^- \oplus G^+ = G$ is a *primary splitting* for $\{T_n\}$ iff the altitudes A_n^-, A_n^+ of T_n respecting $G^- \oplus G^+$ satisfy

$$\operatorname{bol}(A_n^-) \text{ is bounded} \quad \text{as } n \to \infty,$$

$$m(A_n^+)/\|A_n^-\| \to \infty \quad \text{as } n \to \infty.$$

Definition. $m(A^+)/\|A^-\|$ is called the hyperbolicity of T respecting $G = G^- \oplus G^+$.

Definition. Let $\{T_n\}$ be a sequence in $M(G, H)$. Then $\bigoplus_{j=1}^L G^j$ is a *complete splitting* for $\{T_n\}$ iff the altitudes A_n^j of T_n respecting $\bigoplus G^j$ satisfy

$$\operatorname{bol}(A_n^j) \text{ is bounded} \quad \text{as } n \to \infty, \quad 1 \leq j \leq L,$$

$$m(A_n^{j+1})/\|A_n^j\| \to \infty \quad \text{as } n \to \infty, \quad 1 \leq j \leq L-1.$$

Definition. Let F, F' be subspaces of G; let f, f' be the orthogonal projections of G onto F, F'. Then

$$\angle(F, F') = \sup\{\angle(p, f'p): p \in F - 0\}$$

where the angle $(\angle)$ between two vectors means the *acute* angle between the lines they span in G. If one of the vectors is zero, the angle is by definition $\pi/2$.

Thus, $\angle(F, F')$ means: at most how far is F off F'.

Remark. $\angle(F, F') = \angle(F', F)$ if F and F' have the same dimension. It is enough to show $\angle(F, F') \geq \angle(F', F)$. By Euclidean geometry we see that, for $p \in F$, $\angle(f'p, ff'p) \leq \angle(p, f'p)$. We can assume $f'(F) = F'$. For otherwise F contains a vector perpendicular to F' so that $\angle(F, F') = \pi/2$, certainly $\geq \angle(F', F)$. Take any $p' \in F' - 0$. We know there is a (unique) $p \in F - 0$ with $p' = f'p$. Hence $\angle(p', fp') = \angle(f'p, ff'p)$ which by our initial observation is $\leq \angle(p, f'p)$. Hence, $\angle(F', F) \leq \angle(F, F')$, and so they are equal.

Notation. For any vector $x \in G$, we write $\angle(x, F)$ instead of $\angle(\text{span}(x), F)$. For trigonometric functions we shall omit the notation $\angle$. That is, we write $\sin(F, F')$ instead of $\sin(\angle(F, F'))$, etc.

The following lemma simplifies much algebra.

Lemma 0. *Any isomorphism $T: G \to H$ factors uniquely as $T = OP = P'O'$ where O, O' are orthogonal and P, P' are positive definite symmetric.*

Proof [1]. Let $P = ({}^tTT)^{1/2}$, $P' = (T\,{}^tT)^{1/2}$.

3. **Some estimates on altitudes.** We need several geometric inequalities so that we can estimate the norms and bolicities of the altitudes of a monomorphism $T: G \to H$.

Lemma 1. *Suppose $G = F \oplus X$ and Y is a subspace of G with $Y \cap F = 0$. Let $\sigma: Y \to X$ by projection along F. Then*

$$\|\sigma\| \leq 1/\cos(F, X^{\perp}), \qquad m(\sigma) \geq \cos(Y, F^{\perp}).$$

Proof. For any $y \in Y$, $y = f + x$ uniquely, $f \in F$, $x \in X$, and $\sigma y = x$. If $|y| = 1$, trigonometry shows that

$$|f| = \frac{\sin(y, x)}{\sin(f, x)}, \qquad |x| = \frac{\sin(y, f)}{\sin(f, x)}$$

so that $|\sigma y| = \sin(y, f)/\sin(f, x)$. Hence $|\sigma y| \geq \sin(y, F) \geq \min\{\sin(y, F): y \in Y\}$ $= \min\{\cos(y, F^{\perp}): y \in Y\} = \cos(Y, F^{\perp})$ since the minimum of the cosine is achieved for the maximum angle. Similarly $|\sigma y| \leq 1/\cos(F, X^{\perp})$. Q.E.D.

Lemma 2. *Let $T: R^2 \to H$ be a monomorphism. Then*

$$\frac{\sin(Tu, Tu')}{\sin(u, u')} = \frac{|\det(T)|}{|Tu|\,|Tu'|} = \frac{m(T)\|T\|}{|Tu|\,|Tu'|}$$

for any independent unit vectors $u, u' \in R^2$.

Proof. This follows at once from the facts: $|\det(T)|$ = area of the parallelogram spanned by (Tu, Tu')/area of the parallelogram spanned by (u, u'); and $|\det(T)| = m(T)\|T\|$. Q.E.D.

Lemma 3. *Let $G = F \oplus X$ and $T: G \to H$ be a monomorphism. Then the altitude $A: X \to (TF)^{\perp}$ satisfies*

$$\|A\| \le \|T\|, \qquad m(A) \ge m(T)\cos(X, F^{\perp}).$$

Proof. Take any $x \in X$, $|x| = 1$. Then $|Ax| = |Tx|\sin(Tx, TF) = |Tx|\sin(Tx, Tf)$ for some $f \in F$, $|f| = 1$. Call $T' = T|\operatorname{span}(x, f)$. By Lemma 2,

$$|Tx|\sin(Tx, Tf) = \frac{|Tx| m(T')\|T'\|\sin(x, f)}{|Tx|\,|Tf|}.$$

Since $\min \sin(x, f) \ge \min\{\sin(x, F)\} = \cos(X, F^{\perp})$, this fraction is $\ge m(T')\cos(X, F^{\perp}) \ge m(T)\cos(X, F^{\perp})$ as claimed. Also $\max\{\sin(x, f)\} \le 1$ so that $|Ax| \le \|T'\| \le \|T\|$ as claimed. Q.E.D.

Definition. Let $G \subset H$ be a subspace and $0 \le \phi \le \pi/2$ be given. The cone of angle ϕ around G in H is

$$C_{\phi}(G; H) = \{h \in H: h = 0 \text{ or } \angle(h, G) \le \phi\}.$$

and the unit sphere in G is $SG = \{g \in G: |g| = 1\}$.

Lemma 4. *Suppose $P: H \to H$ is an isomorphism, $H = E^- \oplus E^+$, $PE^{\pm} = E^{\pm}$ and $\|P^-\| \le m(P^+)$. Then for all $0 \le \phi \le \pi/2$,*

$$\operatorname{dist}(P(SE^+), P(C_{\phi}E^-)) \ge \|P^-\|/2^{1/2}[(1/b)^2 + (\operatorname{bol}(P^-)\tan\phi)^2]^{1/2} \tag{3.1}$$

where $P^{\pm} = P|E^{\pm}$, b = hyperbolicity of P respecting $E^- \oplus E^+ = H$.

See Figure 1. The factor $2^{1/2}$ may be superfluous.

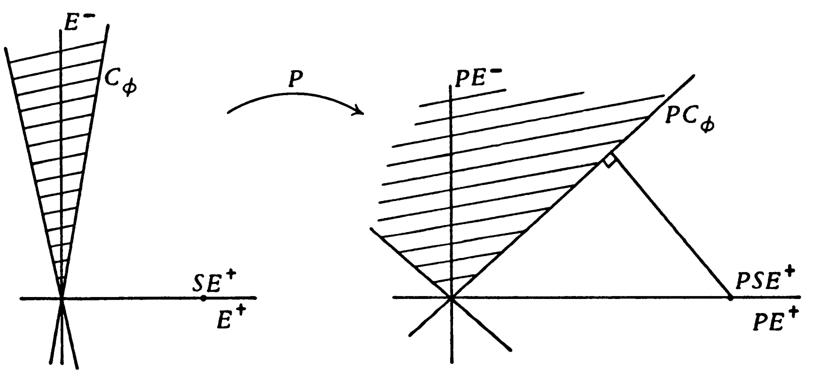

Figure 1

Proof of Lemma 4. For $\phi = \pi/2$ both sides of (3.1) vanish, so we may suppose that $0 \le \phi < \pi/2$. Choose any $g \in SE^+$ and any $c \in C_{\phi}E^-$, $c \ne 0$. Then $c = e^- + e^+$, $e^{\pm} \in E^{\pm}$, and $e^- \ne 0$ since $\phi < \pi/2$ implies $c \notin E^+$. If $e^+ = 0$

then $\operatorname{dist}(Pg, \operatorname{span}(Pc)) = |Pg| \geq m(P^+) \geq \|P^-\| >$ the r.h.s. of (3.1) since $m(P^+) \geq \|P^-\|$ implies $b \geq 1$. Thus we may suppose that $e^+ \neq 0$ and consider $v = e^+/|e^+| \in E^+$.

Let $G = \operatorname{span}(g, v)$, $H' = e^- \oplus G$, and consider $SG \subset SE^+$, $C'_\phi = C_\phi(\operatorname{span}(e^-); H')$. It suffices to prove

$$\operatorname{dist}(P(SG), PC'_\phi) \geq \mu/2^{1/2}[\mu^2/m^2 + \tan^2\phi]^{1/2} \tag{3.2}$$

where $\mu = |Pe^-|/|e^-|$, $m = m(P|G)$. For

$$\mu/[\mu^2 m^{-2} + \tan^2\phi]^{1/2} 2^{1/2} = 2^{-1/2}[m^{-2} + \mu^{-2}\tan^2\phi]^{-1/2}$$

$$\geq 2^{-1/2}[m(P^+)^{-2} + m(P^-)^{-2}\tan^2\phi]^{-1/2} = \text{the r.h.s. of (3.1).}$$

The cone $C_\phi E^-$ and the sphere SE^+ are invariant by orthogonal isomorphisms of H leaving $E^\pm$ invariant. Thus it is no loss of generality to suppose $Pe^- = \mu e^-$, $PG = G$, and $P|G$ is positive definite symmetric since according to Lemma 0 there is an orthogonal $O: H \to H$ having $PO(H') = H'$ and $PO|H'$ positive definite symmetric.

Let e_1, e_2 be unit eigenvectors of $P|G$ having eigenvalues $0 < m \leq M$. If $\dim(G) = 1$ then $e_1 = e_2$ and $m = M$. Otherwise $e_1 \perp e_2$. Call $e_3 = e^-/|e^-|$. Inscribe a square Σ in SG with edges parallel to e_1, e_2. Let $F_1^\pm = e_1 \oplus [e_3 \pm (\tan\phi)e_2]$, $F_2^\pm = e_2 \oplus [e_3 \pm (\tan\phi)e_1]$. These four planes are tangent to C'_ϕ. Their union contains a square cone Σ_ϕ in which C'_ϕ is inscribed. Thus it is clear that $\operatorname{dist}(P(SG), PC'_\phi) \geq \operatorname{dist}(P\Sigma, P\Sigma_\phi)$.

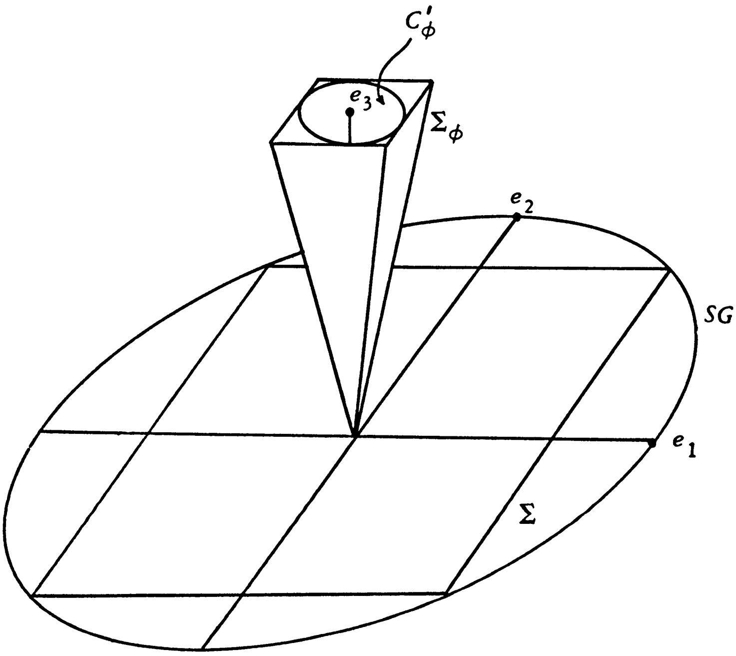

Figure 2

Take any point $s = \pm 2^{-1/2}e_1 + ye_2$, $y \in R$. Then $\operatorname{dist}(Ps, P\Sigma_\phi) = \operatorname{dist}(Ps, PF_2^-) = \operatorname{dist}(Ps, PF_2^+)$ and

$$\operatorname{dist}(Ps, PF_2^+) = \operatorname{dist}(\pm Pe_1, PF_2^+)/2^{1/2} = |Pe_1| \sin(Pe_1, PF_2^+)/2^{1/2}$$
$$= m\mu \cos\phi/2^{1/2}[\mu^2\cos^2\phi + m^2\sin^2\phi]^{1/2}$$

which equals the r.h.s. of (3.2). Similarly, for $s = xe_1 \pm 2^{1/2}e_2$, $x \in R$, we have $\operatorname{dist}(Ps, P\Sigma_\phi) \geq \operatorname{dist}(Ps, PF_1^+)$ and

$$\operatorname{dist}(Ps, PF_1^+) = \mu/2^{1/2}[\mu^2/M^2 + \tan^2\phi]^{1/2}$$

which is greater than or equal to the r.h.s. of (3.2) since $m \leq M$. All points of Σ are of the form $\pm 2^{-1/2}e_1 + ye_2$ or $xe_1 \pm 2^{-1/2}e_2$ so this completes the proof of (3.2) and hence of (3.1). Q.E.D.

Theorem 1. *Let $P: G \to H$ be a monomorphism, $G = E^- \oplus E^+ = G^- \oplus G^+$. Let $A^\pm: G^\pm \to (PG^\mp)^\perp$ be the altitudes. Let b_E, b_G be the hyperbolicities of P respecting $E^- \oplus E^+$, $G^- \oplus G^+$. Suppose $PE^- \perp PE^+$, $m(P^-) = m(P)$, and $\|P^-\| \leq m(P^+)$ for $P^\pm = P|E^\pm$. Then*

(3.3) $$E^- \subset \{x \in G: \sin(x, G^-) \leq \operatorname{bol}(P^-)/b_G\} = C_\phi(G^-; G)$$

for $\phi = \sin^{-1}(\operatorname{bol}(P^-)/b_G)$. Whenever $\sigma: E^- \to G^-$ by projection along G^+ is a well defined monomorphism,

(3.4) $$\|A^-|Y\| \leq \|P^-\|/\cos(g^+, E^+),$$

(3.5) $$|A^-x| \geq m(P^+)\cos(x \oplus G^+, E^+)\cos(G^-, G^{+\perp})$$

for $Y = \sigma E^-$ and $x \in G^-$. Furthermore if $Y = G^-$ then

(3.6) $$m(A^+) \geq \frac{\|P^-\|/\cos(G^-, E^-)}{2^{1/2}[b_E^{-2} + (\operatorname{bol}(P^-)\tan(G^-, E^-))^2]^{1/2}}.$$

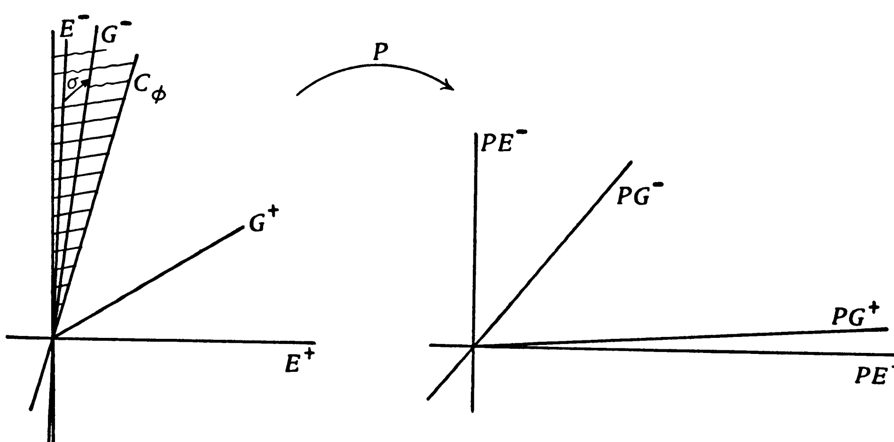

Figure 3

Proof. Take $x \in G$, $|x| = 1$, $x = g^- + g^+$, $g^\pm \in G^\pm$. Then $|g^\pm| = \sin(x, g^\mp)/\sin(g^-, g^+)$ and $A^+g^+ = \pi^+(Pg^+) = \pi^+(Px)$, for $\pi^+: H \to (PG^-)$ by orthogonal projection. Since $\|\pi^+\| = 1$, we have

$$|Px| \geq |\pi^+(Px)| = |A^+g^+| \geq m(A^+)\sin(x, g^-)/\sin(g^-, g^+)$$
$$\geq (m(A^+)/\|A^-\|)\|A^-\|\sin(x, G^-) \geq b_G m(P)\sin(x, G^-)$$

and hence $\{x \in G: |Px|/|x|m(P) \leq K\} \subset \{x \in G: \sin(x, G^-) \leq K/b_G\}$ for any constant K. Since $m(P) = m(P^-)$ and $\sup\{|Px|/|x|m(P^-): x \in E^-\} = \mathrm{bol}(P^-)$ this means that $E^- \subset \{x \in G: |Px|/|x|m(P) \leq \mathrm{bol}(P^-)\} \subset \{x \in G: \sin(x, G^-) \leq \mathrm{bol}(P^-)/b_G\}$ proving (3.3).

That $\|\sigma^{-1}\| \leq 1/\cos(G^+, E^+)$ is Lemma 1. For $y \in Y = \sigma E^-$, $|y| = 1$, $A^-y = \pi^-(P\sigma^{-1}y)$. Hence $|A^-y| \leq \|\pi^-\|\|P^-\|\|\sigma^{-1}\| \leq \|P^-\|/\cos(G^+, E^+)$, proving (3.4).

For $x \in G^-$, $|x| = 1$, Lemma 3 says that $|A^+x| \geq m(P|x \oplus G^+)\cos(x, G^{+\perp})$. For any $g \in x \oplus G^+$, $g = e^- + e^+$, $e^\pm \in E^\pm$, and so

$$|Pg| \geq |Pe^+| \geq m(P^+)\min\{\cos(g, E^+): g \in x \oplus G^+\} = m(P^+)\cos(x \oplus G^+, E^+)$$

as claimed in (3.5).

Suppose $Y = G^-$. Let $\rho: G^+ \to E^+$ by projection along G^-. $\rho G^+ = E^+$ since $Y = G^-$. By Lemma 1, $m(\rho) \geq 1/\cos(G^-, E^-)$. For $x \in G^+$, $A^+x = \pi^+Px = \pi^+P\rho x$ so that $m(A^+) \geq m(\pi^+P^+)m(\rho)$. But

$$m(\pi^+P^+) = \mathrm{dist}(P(SE^+), PG^-) \geq \mathrm{dist}(P(SE^+), PC_{\angle(G^-, E^-)}(E^-)),$$

so Lemma 4 implies (3.6.) Q.E.D.

4. Existence and uniqueness of primary splittings in the positive definite symmetric case.

Theorem 2. *Let $P_k: R^m \to R^m$ be positive definite symmetric, $k = 1, 2, \cdots$. Then there exists a subsequence $\{P_{k_n}\} \subset \{P_k\}$ having a primary splitting $V^- \oplus V^+ = R^m$. That is,*

$$\text{(4.1)} \qquad \begin{aligned} &\text{(a)} \quad \mathrm{bol}(A_n^-) \text{ is uniformly bounded as } n \to \infty, \\ &\text{(b)} \quad m(A_n^+)/\|A_n^-\| \to \infty \quad \text{as } n \to \infty, \end{aligned}$$

where $A_n^\pm: V^\pm \to (P_{k_n}V^\mp)^\perp$ are the altitudes.

Proof. For each P_k there are m orthonormal eigenvectors $e_k^1, \cdots, e_k^m$ and m corresponding positive eigenvalues $\epsilon_k^1, \cdots, \epsilon_k^m$ which we assume to be indexed by increasing size: $0 < \epsilon_k^1 \leq \cdots \leq \epsilon_k^m$.

By abuse of notation we write subscripts n instead of k_n to denote a subsequence.

Select a subsequence $\{P_n\}$ of $\{P_k\}$ so that the ratios $\epsilon_n^i/\epsilon_n^j$ have limits $\mathfrak{S}_j^i \in [0, \infty]$ as $n \to \infty$. Put

$$E_n^- = \mathrm{span}\{e_n^j: \mathfrak{S}_j^1 > 0\}, \qquad E_n^+ = \mathrm{span}\{e_n^j: \mathfrak{S}_j^1 = 0\}.$$

Thus $R^m = E_n^- \oplus E_n^+$ and we can choose a subsequence (not bothering to relabel it) such that $E_n^\pm \to V^\pm$ for some splitting $R^m = V^- \oplus V^+$. This is just the compactness of the Grassmann manifolds and the continuity of the dot product.

Call $P_n^\pm = P_n|E_n^\pm$ and $B = \sup\{\mathrm{bol}(P_n^-)\}$. B is finite by our choice of E_n^- and the fact that P_n is positive definite symmetric. Clearly $m(P_n) = m(P_n^-)$ and so $m(A_n^-) \geq m(P_n^-)$ by Lemma 3. By (3.4), $\|A_n^-\| \leq \|P_n^-\|/\cos(E_n^+, V^+)$. Hence $\mathrm{bol}(A_n^-) = \|A_n^-\|/m(A_n^-) \leq \mathrm{bol}(P_n^-)/\cos(E_n^+, V^+)$ is bounded since $\mathrm{bol}(P_n^-) \leq B$ and $\cos(E_n^+, V^+) \to \cos(V^+, V^+) = 1$. This proves (4.1a).

By (3.4), (3.6), and the definition of B,

$$\frac{m(A_n^+)}{\|A_n^-\|} = \frac{m(A_n^+)}{\|P_n^-\|}\,\frac{\|P_n^-\|}{\|A_n^-\|} \geq \frac{\cos(E_n^+, V^+)}{[b_n^{-2} + B^2\tan^2(E_n^-, V^-)]^{1/2}}$$

for b_n = the hyperbolicity of P_n respecting $E_n^- \oplus E_n^+ = R^m$. This fraction tends to $1/0 = \infty$ as $n \to \infty$ since $b_n \to \infty$ and $E_n^\pm \to V^\pm$. This proves (4.1b). Q.E.D.

Theorem 3. *Let $P_k: R^m \to R^m$ be positive definite symmetric, $k = 1, 2, \cdots$. Let $G^- \oplus G^+ = R^m = W^- \oplus W^+$. Suppose that $W^- \oplus W^+$ is a primary splitting for $\{P_k\}$. Then $G^- \oplus G^+$ is a primary splitting for $\{P_k\} \Leftrightarrow G^- = W^-$.*

Proof. To show "$\Rightarrow$" it is enough to prove that $G^- = V^-$ where $V^\pm$ is as constructed in Theorem 2, $E_n^\pm \to V^\pm$. Let $P_n^\pm = P_n|E_n^\pm$ and let $A_n^\pm: G^\pm \to (P_nG^\mp)^\perp$ be the altitudes. Observe that $\mathrm{bol}(P_n^-)/b_{G_n} \to 0$ since $\mathrm{bol}(P_n^-) \leq B$ and b_{G_n} = the hyperbolicity of P_n respecting $G^- \oplus G^+$, $b_{G_n} \to \infty$ by assumption. Hence (3.3) implies that $\lim(E_n^-) = V^- \subset G^-$.

If $V^- \subsetneqq G^-$ then $G^- \cap E_n^+ \neq 0$ because $\dim(G^-) + \dim(E_n^+) > \dim(V^-) + \dim(E_n^+) = m$. Take $g_n \in G^- \cap E_n^+$, $|g_n| = 1$, $g_n \to g \in G^- \cap V^+$. Applying (3.5) yields

$$\|A_n^-\| \geq |A_n g_n| \geq m(P_n^+)\cos(g_n \oplus G^+, E_n^+)\cos(G^-, G^{+\perp}).$$

By (3.4), $m(A_n^-) \leq \|P_n^-\|/\cos(G^+, E_n^+)$. Hence

$$\frac{\|A_n^-\|}{m(A_n^-)} \geq \frac{m(P_n^+)}{\|P_n^-\|}\cos(g_n \oplus G^+, E_n^+)\cos(G^-, G^{+\perp})\cos(G^+, E_n^+).$$

The product of the three cosines tends to $\cos(g \oplus G^+, V^+)\cos(G^-, G^{+\perp})\cos(G^+, V^+)$ as $n \to \infty$. This is not zero since it is easily seen that $g \oplus G^+$ is independent of V^-, $G^- \cap G^+ = 0$, and $G^+ \cap V^- = 0$. So we have $\|A_n^-\|/m(A_n^-) \to \infty$ which contradicts the assumption that $\{P_k\}$ obeyed (4.1a) respecting $G^- \oplus G^+$. Hence $V^- = G^-$, completing the proof of "$\Rightarrow$".

Now suppose that $G^- = W^-$. By "$\Rightarrow$" $W^- = V^-$ for $E_n^- \to V^-$. Again let $A_n^\pm: G^\pm \to (P_nG^\mp)^\perp$ be the altitudes. By (3.4) and Lemma 3

$$\frac{\|A_n^-\|}{m(A_n^-)} \le \frac{\|P_n^-\|}{m(P_n)\cos(G^+, E_n^+)} = \frac{\|P_n^-\|}{m(P_n^-)\cos(G^+, E_n^+)} = \frac{\operatorname{bol}(P_n^-)}{\cos(G^+, E_n^+)}.$$

But $\operatorname{bol}(P_n^-) \le B$ and $\cos(G^+, E_n^+) \longrightarrow \cos(G^+, V^+) \ne 0$ since $V^- \oplus V^+ = R^m = G^- \oplus G^+$ and $G^- = V^-$. Hence $\{A_n^-\}$ obeys (4.1a).

As for (4.1b), by (3.4) and (3.6) we have

$$\frac{m(A_n^+)}{\|A_n^-\|} = \frac{m(A_n^+)\|P_n^-\|}{\|P_n^-\|\,\|A_n^-\|} \ge \frac{\cos(G^+, E_n^+)/\cos(G^-, E_n^-)}{[b_{E_n}^{-2} + (\operatorname{bol}(P_n^-)\tan(G^-, E_n^-))^2]^{1/2}}$$

where b_{E_n} = hyperbolicity of P_n respecting $E_n^- \oplus E_n^+$. But $b_{E_n} \longrightarrow \infty$, $\operatorname{bol}(P_n^-) \le B$, and $\tan^2(G^-, E_n^-) \longrightarrow 0$ while $\cos(G^+, E_n^+) \longrightarrow \cos(G^+, V^+) \ne 0$ and $\cos(G^-, E_n^-) \longrightarrow 1$. This verifies (4.1b) for $\{A_n^\pm\}$. Q.E.D.

5. Existence of complete splittings.

Existence theorem. *Let $T_k\colon R^m \longrightarrow H$ be a monomorphism, $k = 1, 2, \cdots$. Then there exists a subsequence $\{T_{k_n}\} \subset \{T_k\}$ having a complete splitting: $\bigoplus_{j=1}^L V^j$. That is,*

(5.1)
(a) $\operatorname{bol}(A_n^j)$ *is bounded as* $n \longrightarrow \infty$, $1 \le j \le L$,
(b) $m(A_n^{j+1})/\|A_n^j\| \longrightarrow \infty$ *as* $n \longrightarrow \infty$, $1 \le j \le L-1$,

where $A_n^j\colon V^j \longrightarrow [T_{k_n}({}^jV)]^\perp$ are the altitudes respecting $\bigoplus_{j=1}^L V^j$ and ${}^jV = \bigoplus_{i \ne j} V^i$.

In the proof of the above we need the following lemma

Lemma 5. *If $E \oplus F = G \subset H$ then*

$$\begin{tikzcd} H \arrow[r, "\pi"] \arrow[d, "r"'] & G^\perp \arrow[d, hook] \\ E^\perp \arrow[r, "\omega"] \arrow[ur, "\chi"] & F^\perp \end{tikzcd}$$

commutes, all the maps except the inclusion being orthogonal projections.

Proof. This is clear. Any $x \in H$ is $x = y + e + f$ for some $y \in G^\perp$, $e \in E$, and $f \in F$. Then $r(x) = y + f$, $\omega(rx) = y$, $\chi(rx) = y$, while $\pi(x) = y$.

Proof of the existence theorem. We use induction on m. For $m = 1$, the theorem is trivial since L must be 1 and $m(A_n^j) = \|A_n^j\|$.

Next we observe that if $\bigoplus_{j=1}^L V^j = R^m$ is a complete splitting for $\{T_{k_n}\}$ then it is also one for $\{S_n T_{k_n}\}$ where $\{S_n\}$ is any sequence of orthogonal maps of H. For the altitudes of $S_n T_{k_n}$ are B_n^j where

$$
\begin{tikzcd}
 & H \arrow[r, "\tau^j_n"] & {[S_nT_{k_n}({}^jV)]^\perp} \\
V^j \arrow[ur, "S_nT_{k_n}"] \arrow[dr, "T_{k_n}"'] \arrow[urr, bend left, "B^j_n"] \arrow[drr, bend right, "A^j_n"'] & & \\
 & H \arrow[uu, "S_n"] \arrow[r, "\pi^j_n"] & {[T_{k_n}({}^jV)]^\perp} \arrow[uu, "S_n"']
\end{tikzcd}
$$

π^j_n and τ^j_n being the orthogonal projections. This commutes because S_n is orthogonal. Thus $B^j_n = S_n A^j_n$ and so (5.1) is clear for $\{B^j_n\}$.

By Lemma 0, we may factor each T_k in the given sequence as $T_k = O_k P_k$, $1, 2, \cdots$, where O_k is orthogonal and P_k: $R^m \to R^m$ is positive definite symmetric. According to the preceding paragraph, we need only prove the theorem for the arbitrary sequence $\{P_k\}$.

Again we use n instead of k_n for the subscripts of the subsequence. According to Theorem 2 there is a subsequence $\{P_n\} \subset \{P_k\}$ having a primary splitting $V^- \oplus V^+$ with altitudes $A^\pm_n$: $V^\pm \to [P_n(V^\mp)]^\perp$. If $V^+ = 0$ then (4.1a) implies (5.1a) for $R^m = V^- \stackrel{\text{def}}{=} V^1$ while (5.1b) is vacuous. $V^- \neq 0$ by construction. Thus we may apply the induction hypothesis to the sequence of monomorphisms A^+_n: $V^+ \to H$. This gives a subsequence (unrelabeled) of $\{A^+_n\}$ having a complete splitting $\bigoplus_{j=2}^L V^j = V^+$. Let C^j_n be the altitudes of A^+_n respecting this splitting, $2 \leq j \leq L$.

With $V^- = V^1$, consider the splitting $R^m = \bigoplus_{j=1}^L V^j$ having the altitudes A^j_n: $V^j \to [P_n({}^jV)]^\perp$. Thus $A^1_n = A^-_n$. We claim that $C^j_n = A^j_n$, $j \geq 2$. This holds because the following diagram commutes.

$$
\begin{tikzcd}
 & R^m \arrow[r, "P_n"] & R^m \arrow[r, "\pi^j_n"] \arrow[dd, "\pi^+_n"] & {[P_n({}^jV)]^\perp} \arrow[dd, hook] \\
V^j \arrow[ur, hook] \arrow[dr, hook] \arrow[urrr, bend left, "A^j_n"] \arrow[drrr, bend right, "C^j_n"'] & & & \\
 & V^+ \arrow[uu, hook] \arrow[r, "A^+_n"] & (P_nV^-)^\perp \arrow[r, "\omega^j_n"] & {[\bigoplus_{2\leq i\neq j} A^+_n V^i]^\perp}
\end{tikzcd}
$$

π_n^+, π_n^j, ω_n^j being the orthogonal projections. That there is an inclusion for the rightmost map is true because $A_n^+ V^i$ is just $P_n V^i$ made orthogonal to $P_n V^1$ and so $A_n^+ V^i \subset P_n V^1 \oplus P_n V^i$, $i = 2, \cdots, L$. Thus $\bigoplus_{2 \le i \ne j} A_n^+ V^i \subset P_n({}^j V)$ and the reverse is true for their orthogonal complements.

That the leftmost triangle commutes is trivial. That the first square commutes is the definition of A_n^+. That the second square commutes is Lemma 5 applied to $E = P_n V^-$, $F = \bigoplus_{2 \le i \ne j} A_n^+ V^i$, $G = E \oplus F = P_n({}^j V)$, $H = R^m$.

Thus, $C_n^j = A_n^j$, $j \ge 2$, and so (5.1) holds for $\{A_n^j\}$, $j \ge 2$, while (4.1a) implies (5.1a) for $j = 1$. As for (5.1b) in the case $j = 1$, we have

$$\frac{\|A_n^1\|}{\|A_n^2\|} = \frac{\|A_n^1\|}{m(A_n^+)} \frac{m(A_n^+)}{m(A_n^2)} \frac{m(A_n^2)}{\|A_n^2\|}.$$

The first factor tends to 0 by (4.1b). The second factor is ≤ 1 by Lemma 3. The last factor is always ≤ 1. Hence (5.1b) is verified for $j = 1$, also, completing the proof of the existence theorem. Q.E.D.

6. Uniqueness of complete splittings.

Uniqueness theorem. *Let T_k: $R^m \to H$ be a monomorphism, $k = 1, 2, \cdots$. Suppose that $\{T_k\}$ has the complete splitting $\bigoplus W^j = R^m$. Then $\bigoplus G^j$ is a complete splitting for $\{T_k\} \Longleftrightarrow$ the flags of $\bigoplus G^j$ and $\bigoplus W^j$ are equal.*

Again we need a lemma.

Lemma 6. *If $\bigoplus G^j$ is a complete splitting of a sequence of monomorphisms $G \to H$, $\{T_n\}$, and S: $F \to G$ is an isomorphism then $\bigoplus S^{-1} G^j$ is a complete splitting of $\{T_n S\}$.*

Proof. Let the altitudes of $\{T_n\}$ respecting $\bigoplus G^j$ be A_n^j and of $\{T_n S\}$ respecting $\bigoplus S^{-1} G^j$ be B_n^j. It is clear that $B_n^j = A_n^j S$. Hence $\bigoplus S^{-1} G^j$ is a complete splitting for $\{T_n S\}$. Q.E.D.

Proof of the uniqueness theorem. To make the induction easier, we also assert

$$m(D_n^1)/m(T_n) \text{ is bounded as } n \to \infty \tag{6.1}$$

for D_n^1 = the first altitude of any complete splitting of $\{T_n\}$.

As before, the case $m = 1$ is trivial and it is no loss of generality to replace $\{T_k\}$ with $\{P_n\}$, P_n: $R^m \to R^m$ being positive definite symmetric. Let $\bigoplus V^j = R^m$ be the complete splitting of (an unrelabeled subsequence of) $\{P_n\}$ as constructed in §5. Thus $V^- \oplus V^+$ is a primary splitting for $\{P_n\}$, $V^- = V^1$, $V^+ = \bigoplus_{j \ge 2} V^j$. To prove "$\Rightarrow$" it is enough to prove $\operatorname{flag}(\bigoplus G^j) = \operatorname{flag}(\bigoplus V^j)$.

Let $G^- = G^1$, $G^+ = \bigoplus_{j \ge 2} G^j$, and $A_n^\pm$, A_n^j, $B_n^\pm$, B_n^j the altitudes of P_n respecting $V^- \oplus V^+$, $\bigoplus V^j$, $G^- \oplus G^+$, $\perp G^j$. Thus, $A_n^- = A_n^1$, $B_n^- = B_n^1$. As in

§5, the altitudes of $\{B_n^+\}$ respecting $G^+ = \bigoplus_{j\geq 2} G^j$ are B_n^j, $j \geq 2$. This makes $\bigoplus_{j\geq 2} G^j$ a complete splitting of $\{B_n^+\}$ and so by induction on (6.1), $m(B_n^2)/m(B_n^+)$ is bounded as $n \longrightarrow \infty$. Hence

$$\frac{m(B_n^+)}{\|B_n^-\|} = \frac{m(B_n^+)}{m(B_n^2)} \frac{m(B_n^2)}{\|B_n^1\|} \longrightarrow \infty \quad \text{as } n \longrightarrow \infty.$$

Thus, $G^- \oplus G^+$ is a primary splitting of $\{P_n\}$ and so by Theorem 3, $G^- = V^-$, i.e. $G^1 = V^1$.

Let $\sigma\colon R^m \longrightarrow V^+$ by orthogonal projection (i.e., projection along V^-). Since $G^- = V^-$, it is clear that $A_n^+\sigma = B_n^+$. By Lemma 6, $\bigoplus_{j\geq 2} \sigma^{-1}V^j$ is a complete splitting of $\{A_n^+\sigma\} = \{B_n^+\}$. By induction, $\operatorname{flag}(\bigoplus_{j\geq 2} G^j) = \operatorname{flag}(\bigoplus_{j\geq 2} \sigma^{-1}V^j)$ and hence $\operatorname{flag}(\bigoplus G^j) = \operatorname{flag}(\bigoplus V^j)$.

Now assume that $\operatorname{flag}(\bigoplus G^j) = \operatorname{flag}(\bigoplus W^j)$. By "$\Longrightarrow$" $\operatorname{flag}(\bigoplus W^j) = \operatorname{flag}(\bigoplus V^j)$ for $\bigoplus V^j$ as above. In particular, $V^1 = G^1$ so that $\sigma\colon G^+ \longrightarrow V^+$ as in the proof of "$\Longrightarrow$" is well defined and $\operatorname{flag}(\bigoplus_{j\geq 2} \sigma^{-1}V^j) = \operatorname{flag}(\bigoplus_{j\geq 2} G^j)$. By Lemma 6, $\bigoplus_{j\geq 2} \sigma^{-1}V^j$ is a complete splitting of $\{A_n^+\sigma\} = \{B_n^+\}$. By induction, $\bigoplus_{j\geq 2} G^j$ is a complete splitting of $\{B_n^+\}$. The altitudes of B_n^+ respecting $\bigoplus_{j\geq 2} G^j$ are B_n^j, $j \geq 2$, as shown above. Also as above,

$$m(B_n^2)/m(B_n^+) \text{ is bounded} \quad \text{as } n \longrightarrow \infty. \tag{6.2}$$

Thus, (5.1a) holds for B_n^j, $j \geq 2$, and (5.1b) holds for $j \geq 2$. $V^- \oplus V^+$ is a primary splitting of $\{P_n\}$ by construction. By Theorem 3 and $G^- = V^-$, $G^- \oplus G^+$ is also a primary splitting of $\{P_n\}$. Hence (5.1a) also holds for $B_n^1 = B_n^-$ and $m(B_n^+)/\|B_n^-\| \longrightarrow \infty$ as $n \longrightarrow \infty$. With (6.2) this implies that

$$\frac{m(B_n^2)}{\|B_n^-\|} = \frac{m(B_n^2)}{m(B_n^+)} \frac{m(B_n^+)}{\|B_n^-\|} \longrightarrow \infty \quad \text{as } n \longrightarrow \infty.$$

That is, (5.1b) also holds for $j = 1$, which completes the proof of "$\Longleftarrow$".

All that remains to prove is (6.1) in dimension m. Again it is no loss of generality to replace $\{T_n\}$ with $\{P_n\}$, $P_n\colon R^m \longrightarrow R^m$ being positive definite symmetric. For $m(D_n^1)/m(T_n)$ is unchanged when we multiply T_n on the left by an orthogonal isomorphism of H. So let F^j be any complete splitting of $\{P_n\}$, $D_n^j\colon F^j \longrightarrow (P_n{}^jF)^\perp$ the altitudes.

By the first half of the proof of this theorem, $F^1 = V^1$ where $\bigoplus V^j = R^m$ is as above. By construction of $\bigoplus V^j$, $V^1 = \lim(E_n^-)$ where $E_n^- \oplus E_n^+ = R^m$, $P_nE_n^- \perp P_nE_n^+$, and $m(P_n^-) = m(P_n)$ for $P_n^\pm = P_n|E_n^\pm$. Thus, Theorem 3 applies.

By (3.4), $\|D_n^1\| \leq \|P_n^-\|/\cos(F^+, E_n^+)$ and so

$$\frac{m(D_n^1)}{m(P_n)} \le \frac{\|D_n^1\|}{m(P_n^-)} \le \frac{1}{\cos(F^+, E_n^+)} \longrightarrow \frac{1}{\cos(F^+, V^+)} < \infty$$

completing the proof of (6.1) in dimension m, also. Q.E.D.

7. **An example.** Let $\{P_n\}$ be a sequence of positive definite symmetric isomorphisms $R^m \to R^m$ whose eigenvalues are $0 < \epsilon_n^1 \le \cdots \le \epsilon_n^m$. Let $e_n^1, \cdots, e_n^m$ be a corresponding orthonormal set of eigenvectors. Suppose $\mathfrak{E}_j^i = \lim_n \epsilon_n^i/\epsilon_n^j$ exist, for each i, j, $\mathfrak{E}_j^i \in [0, \infty]$. Because we ordered the ϵ^i by increasing size, we have: $\mathfrak{E}_j^i$ increases as i increases, $\mathfrak{E}_j^i$ decreases as j increases, $\mathfrak{E}_i^i = 1$. Consequently, the finite nonzero entries of the matrix $\mathfrak{E}_j^i$ give a sequence of blocks down the diagonal. Partition the eigenvectors according to these blocks. That is, we say that ϵ_n^i is equivalent to ϵ_n^j iff $\mathfrak{E}_j^i$ is nonzero, finite. Then we look at the equivalence classes of the ϵ_n^i and the corresponding classes of the e_n^j. There are say L of them, $L \le m$. We look at the space spanned by all the e_n^j equivalent to a given one. Letting $j = 1, \cdots, m$, this gives L distinct orthogonal subspaces, $E_n^1, \cdots, E_n^L$. Their dimensions are independent of n.

Suppose $E_n^j \xrightarrow[n]{} V^j$, $1 \le j \le L$. (This could be arranged by a subsequence of course.) It would be natural to expect that $\bigoplus V^j = R^m$ is a complete splitting of $\{P_n\}$. This is not true. It was really necessary to go first to $R^m = V^- \oplus V^+$, then to $V^+ = \bigoplus_{j=2}^L V^j$ as in the proof of the existence theorem.

In fact, consider, for positive parameters ϵ and δ, the 3×3 matrix

$$P = \begin{pmatrix} \delta + (1-\delta)\sin^2(\epsilon) & (1-\delta)\sin(2\epsilon)/2 & 0 \\ (1-\delta)\sin(2\epsilon)/2 & \delta + (1-\delta)\cos^2(\epsilon) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This makes $P = R_{-\epsilon} Q_\delta R_\epsilon$ where

$$Q_\delta = \begin{pmatrix} \delta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_\epsilon = \begin{pmatrix} \cos(\epsilon) & -\sin(\epsilon) & 0 \\ \sin(\epsilon) & \cos(\epsilon) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

R_ϵ is the rotation through angle ϵ in the (x, y)-plane. The eigenspaces of P are the images by $R_{-\epsilon}$ of those of Q_δ. Thus they are $E^1 = R_{-\epsilon}(x\text{-axis})$, $E^2 = R_{-\epsilon}((y, z)\text{-plane})$. As $\epsilon \to 0$, $E^1 \to V^1 = x$-axis and $E^2 \to V^2 = (y, z)$-plane.

We let ϵ and δ tend to zero. This makes E^1 and E^2 spaces of equivalent vectors under the notion of equivalence defined above. We show, however, that if $\epsilon/\delta \to \infty$ while $\epsilon, \delta \to 0$, then $A^2(z) \equiv z$ and $A^2(y) \to 0$ for $z = (0, 0, 1)$, $y = (0, 1, 0)$. This means that (5.1a) is violated for the splitting $R^3 = V^1 \oplus V^2$. That $A^2(z) \equiv z$ is clear.

Call $x = (1, 0, 0)$. Then $|A^2 y| = |Py| \sin(Px, Py)$. By Lemma 2 and inspection

$(\delta \leq 1)$, $\sin(Px, Py) = m(P')\|P'\|/|Px||Py| = \delta/|Px||Py|$, where $P' = P|\mathrm{span}(x, y)$. Hence

$$|A^2 y| = \delta/|Px| = \delta/|(\delta + (1-\delta)\sin^2(\epsilon), (1-\delta)\sin(2\epsilon)/2)|$$
$$= |(1 + ((1-\delta)/\delta)\sin^2(\epsilon), ((1-\delta)/2\delta)\sin(2\epsilon))|^{-1}$$

which tends to zero as $\epsilon, \delta \longrightarrow 0$ and $\epsilon/\delta \longrightarrow \infty$. The correct splitting is (x-axis) $\oplus$ (y-axis) $\oplus$ (z-axis) as is easily checked.

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